An Iterative Method for Solving Partial Differential Equations and Solution of Korteweg-de Vries Equations for Showing the Capability of the Iterative Method

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Abstract: In this paper demonstrated an iterative method as a semi-analytical method for solving nonlinear partial differential equations. Many non-linear PDE that arising in various engineering applications have no exact solution so a semi-analytical solution of these equations is very important. Here has been considered some nonlinear partial differential equations with appropriate initial/boundary conditions and clearly demonstrated the capability of the method. Then the Korteweg-de Vries equations are solved by iterative method. The results shown that the iterative method is accurate, convenient, fast, time saver and have a high order of convergence and have no need to the restriction assumptions and small perturbation parameter.

Keywords: An iterative method, PDE, semi-Analytical solution, Korteweg-de Vries equations, kdv

I. INTRODUCTION

There are some non-linear partial differential equations (PDE) in the solving of many physical problems that solution of these problems is very important. Despite of many efforts that have been done for obtaining the exact solution of non-linear partial differential equations, so far not presented an exact solution for many problems. The researchers are looking for semi-exact and analytical methods.

The Adomian’s decomposition method (ADM) [1] was presented in 1988 by Adomian’s. This method had been introduced a semi-exact and approximation solution for partial differential equations (PDE’s) and ordinary differential equations (ODE’s). For years the researchers were using of this method for solution of PDE and ODE. Many researchers after 1988 improved this method until Ji-Huan He presented the homotopy perturbation method (HPM) [2-7] in 1998. The benefits of this method relative to past methods is better accurate, little computational volume, easy using, etc.

There are many standard methods for solving nonlinear partial differential equations [8]; for instance: Backland transformation method [9], Lie group method [10], inverse scattering method [11], Hirota’s bilinear method [12], homogeneous balance method [13], VIM that was proposed by He[14-18]; H. Temim and A.R. Ansari introduced a semi-analytical iterative technique for solving nonlinear ordinary differential equations[19] but The goal of this paper is presentation of a semi-exact and analytical method for solving partial differential equations.

Despite of past methods iterative method is easier to use, has more accurate, no need restriction assumptions, has little computational volume. Another advantage of iterative method is on the provision of a rapidly convergent series and after fewer iterative the results will be converged and could be approached the close form of exact solution.

II. BASIC IDEA OF ITERATIVE METHOD

To illustrate the basic concepts of iterative method, considering the following partial differential equation

\[ A(u(x,t)) - f(x,t) = 0 \quad r \in \Omega \] (1)

Subject to boundary condition

\[ B(u, \frac{\partial u}{\partial n}) = 0 \quad x \in \Gamma \] (2)
Where $A$ is a general differential operator, $B$ a boundary operator $f(x)$ is a known analytical function, $\Gamma$ is the boundary of domain $\Omega$ and $\frac{\partial u}{\partial n}$ denotes differentiation along the normal drawn outwards from $\Omega$. The operator $A$ can, generally speaking, be divided into two parts: a linear part $L$ and a nonlinear part $N$. Eq. (1) therefore can be rewritten as follows:

$$L(u(x,t)) + N(u(x,t)) - f(x,t) = 0$$

(3)

In this method by the elimination of non-linear term, the initial guess is obtained as follows

$$L(u_0(x,t)) = f(x,t) \quad B(u_0, \frac{\partial u_0}{\partial n}) = 0 \quad x \in \Gamma$$

(4)

Then by replacing the $u_n(x,t)$ and $u_{n+1}(x,t)$ respectively in the non-linear and linear term of PDE, next iterations obtained as Eq. (5)

$$L(u_{n+1}(x,t)) = -N(u_n(x,t)) + f(x,t)$$

(5)

In fact each $u_i(x,t)$ is separately the solution of problem. The iterative method is easy to use and each solution is a betterment of the previous iteration, Continuing this manner, until could be obtained $u_n(x,t)$ which is in the good approximate with the exact solution.

### III. APPLICATION OF ITERATIVE METHOD

#### III.1 Example 1. Solution Korteweg-de Vries (kdv) equation by iterative method

Considering the following K (2, 2) equation:

$$u_t + (u^2)_x + (u^3)_{xxx} = 0$$

(6)

With the following initial condition:

$$u(x,0) = x$$

(7)

Exact solution $\frac{x}{1 + 2t}$

(8)

For solution of Eq. (6) with initial condition (7) it is used of iterative method:

$$L(u(x,t)) = u_t$$

(9)

$$N(u(x,t)) = (u^2)_x + (u^3)_{xxx}$$

(10)

$$f(x,t) = 0$$

(11)

$$\frac{\partial u_0}{\partial t} = 0, u_0(x,0) = u_o = x$$

(12)

$$\frac{\partial u_{n+1}}{\partial t} = -\left(\frac{\partial u_n^2}{\partial x}\right) - \frac{\partial^3 u_n^2}{\partial x^3}, u_{n+1}(x,0) = x$$

(13)

By assuming above initial conditions a solution for equation (6) by iterative method is as follows:

$$u_0 = x$$

(14)

$$u_1 = x(1-2t)$$

(15)

$$u_2 = x\left(1-2t+4t^2 - \frac{8t^3}{3}\right)$$

(16)

$$u_3 = x\left(1-2t+4t^2 - 8t^3 + \frac{32}{3}t^4 - \frac{32}{3}t^3 + \frac{64}{9}t^6 - \frac{128}{63}t^7\right)$$

(17)
Each of above terms is the solution of equation (6) but must be choose the best answer by comparing $u_{n-1}$ and $u_n$

Until that $|u_n - u_{n-1}| \leq e$, $e = \text{computational accurate}$

Here are written more iteration for better solution; although next iteration is better that previous iteration. Everyone can probe this statement by comparing each $u_n$ with exact solution alone. In this example after five iterations, it could be obtain the analytical solution (19) as a close form of exact solution

Obviously it is shown that $u_n$ converges to $\frac{x}{1+2t}$ as an exact solution

**III.2 Example 2. Solution Korteweg-de Vries (kdv) by iterative method**

Considering the KdV equation as:

\[ u_t + \frac{1}{2}(u^2)_x - u_{xxx} = 0 \]  

(20)

With the following initial condition:

\[ u(x,0) = x \]  

(21)

Exact solution $= \frac{x}{1+2t}$

To solve Eq (20) with initial condition (21) there is used of iterative method that the correction functional defined as:

\[ L(u(x,t)) = u_t \]  

(22)

\[ N(u(x,t)) = \frac{1}{2}(u^2)_x - u_{xxx} \]  

(23)

\[ f(x,t) = 0 \]  

(24)

\[ \left( \frac{\partial u}{\partial t} \right) = 0, u_0(x,0) = u_0 = x \]  

(25)
\begin{equation}
\left( \frac{\partial u_{n+1}}{\partial t} \right) = -\frac{1}{2} \left( \frac{\partial u_n}{\partial x} \right) + \frac{\partial^2 u_n}{\partial x^2}, \quad u_{n+1}(x,0) = x
\end{equation}

By assuming above initial conditions a solution for equation (20) by iterative method is as follows:

\begin{align*}
u_0 &= x \\
u_1 &= x(1-t) \\
u_2 &= x \left[ 1-t + t^2 - \frac{1}{3} t^3 \right] \\
u_3 &= x \left[ 1-t + t^2 - t^3 + \frac{2}{3} t^4 - \frac{1}{3} t^5 + \frac{1}{9} t^6 - \frac{1}{63} t^7 \right] \\
u_4 &= x \left[ 1-t + t^2 - t^3 + t^5 - \frac{13}{15} t^7 + \frac{2}{3} t^8 - \frac{1}{63} t^9 + \frac{71}{252} t^{10} - \frac{86}{567} t^{11} + \frac{1}{315} t^{12} \right] \\
u_5 &= x \left[ 1-t + t^2 - t^3 + t^5 - \frac{13}{15} t^7 + \frac{2}{3} t^8 - \frac{1}{63} t^9 + \frac{71}{252} t^{10} - \frac{86}{567} t^{11} + \frac{1}{315} t^{12} - \frac{5}{189} t^{13} + \frac{1}{567} t^{14} - \frac{1}{3969} t^{15} + \frac{1}{59535} t^{16} \right]
\end{align*}

After trying the higher iterations, the exact solution could be obtained. Fortunately, in this example after four iterations, the analytical solution (32) obtained that was at a close form of exact solution, although \( u_4 \) could be the solution but for the little error must be choice \( u_5 \) as the analytical solution.

\section*{IV. Conclusion and Results}

Table (1) and (2) show the difference between iterative method and exact solution in solving the example 1 and example 2 for different values of \( x \) and \( t \). In the figures (1) and (2), two-dimensional plots are shown for the comparison of each iteration that it is obtained from iterative method with exact solution for different values of \( x \) and \( t=0.6 \). In the end at the figure (3) \( u(x,t) = u_4 \) and \( u(x,t) = u_5 \) are shown by three-dimensional plots for \( 0 \leq t \leq 1 \), \( 0 \leq x \leq 3 \) also the results converged to the \( u(x,t) = \frac{x}{1+2t} \) after sentence 5 of iterative method in example 1.

At Figure (4) is shown \( u(x,t) = u_3 \), \( u(x,t) = u_4 \) and \( u(x,t) = \frac{x}{1+2t} \) for \( 0 \leq t \leq 1 \), \( 0 \leq x \leq 3 \) at the example 2.

In this paper, iterative method has been successfully applied to find the solution of kdv equations. Then the solution is compared with exact solution that shown the iterative method has the best accurate against of past methods. The best advantage of iterative method is it's simplify and accurate. The results of the iterative method are in approximately agreement with exact solution. The iterative method is capable to solve other partial differential equations and it can be used instead of other methods for finding semi-analytical solution of PDE.
Table 1. Difference between the iterative method and exact solution of KdV equation in the example 1 for different values of x, t

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<th>t</th>
<th>NIM</th>
<th>EXACT</th>
<th>t</th>
<th>NIM</th>
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Table 2. Difference between the iterative method and exact solution of KdV equation in the example 2 for different values of x, t

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Figure 1. The comparison of Exact solution and $u_1, u_2, u_3, u_4, u_5$ of example 1 for different values of x and $t = 0.6$
Figure 2. The comparison of Exact solution and $u_1, u_2, u_3, u_4$ of example 2 for different values of $x$ and $t = 0.6$.

Figure 3. Three-dimensional plot for the solution obtained by (a) $u_4$, (b) $u_5$, (c) Exact solution.
(\(u_3\) and \(u_4\) are the terms of iterative method in the example 1)

(a) 

(b) 

(c)

Figure 4. Three-dimensional plot for the solution obtained by (a) \(u_3\), (b) \(u_4\), (c) Exact solution

(\(u_3\) and \(u_4\) are the terms of iterative method in the example 2)

REFERENCE


